Chern-Simons Lecture Series 20131202 ~ 06

Plan

1. a) Hilbert scheme of points on a smooth complex surface and Heisenberg algebras b) Virasoro algebras

2. a) Uhlenbeck spaces
   b) W-algebras

3. Instantons and W-algebras (with Braverman + Finkelberg)

§0. motivation

AGT conjecture
Alday-Gaiotto-Tachikawa

4d gauge theory $\longleftrightarrow$ 2d CFT

duality come from

6^d theory cptified on $X^4 \times C^2$

$X = \mathbb{R}^4$ + T^2-action $\longleftrightarrow$ W-algebra

equivariant parameters $\varepsilon_1, \varepsilon_2$ level k

$$\boxed{k + h^\vee = -\frac{\varepsilon_2}{\varepsilon_1}}$$

Unfortunately 6d theory is difficult to be justified in mathematically rigorous way.

Therefore we take a down-to-earth approach :
We construct a W-algebra representation on $IH^*_{G\times T^2}$ (instanton moduli sp.)

generalization of my old work on Hilb sch. & Heis. alg.

This is only the Hilbert space attached to the bdry of C .
It is still a long way to construct the actual CFT.

§1a

X : smooth complex surface,
assume X : projective for a while

$X^{[n]}$ = Hilbert scheme of n points on X
$\pi \searrow$
$S^n X = X^n / \mathfrak{S}_n$: symmetric power

<u>Facts</u>
1) $X^{[n]}$: smooth, connected, $\dim = 2n$ (Fogarty)
2) $X^{[n]}$: symplectic if X : symplectic (Beauville)
3) $\mathcal{B}^{[n]} := \{ (Z, x) \in X^{[n]} \times X \mid \pi(Z) = nx \} \xrightarrow{p} X$ (Briançon)
irreducible, $\dim = \frac{1}{2} \dim X^{[n]} \times X = n+1$ fiber bdle

∘ stratification
$$S^n X = \coprod_{\lambda \vdash n} S_\lambda X$$
$S_{(1^n)} X$: distinct points

$S_{(n)} X$: a single pt with multiplicity n

— π : isom. on $\pi^{-1}(S_{(1^n)} X)$
— $n=2$ $\quad X^{[2]} = \pi^{-1}(S_{(1^2)} X) \sqcup \underbrace{\pi^{-1}(S_{(2)} X)}_{= \mathcal{B}^{[2]}}$ $\quad v \in \mathbb{P}(T_x X)$

$x \in X$

Consider $\displaystyle\bigoplus_{n=0}^{\infty} H^*(X^{[n]})$

<u>$m>0$</u>

may impose $Z_1 \supset Z_2$

$$X^{[n]} \times X^{[n+m]} \times X \supset \mathcal{P}^{[n,n+m]} := \{(Z_1, Z_2, x) \mid \pi(Z_2) = \pi(Z_1) + mx\}$$

$$\dim = \tfrac{1}{2}(\text{total dim}) = 2n+m+1$$

$$\begin{tikzcd}
& X^{[n]} \times X^{[n+m]} \times X \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "\pi"] & \\
X^{[n]} & X^{[n+m]} & X
\end{tikzcd}$$

$\alpha \in H^*(X)$

$$P_m(\alpha) : H^*(X^{[n+m]}) \longrightarrow H^*(X^{[n]})$$
$$c \longmapsto p_{1*}\left(p_2^*(c) \cup \pi^*(\alpha) \cap [\mathcal{P}^{[n,n+m]}]\right)$$

$$P_{-m}(\alpha) : H^*(X^{[n+m]}) \longrightarrow H^*(X^{[n]})$$
$$d \longmapsto \pm\left(p_{2*}(p_1^*(d) \cup \pi^*(\alpha) \cap [\mathcal{P}^{[n,n+m]}]\right)$$

<u>sign convention</u>

$H^*(X^{[n]})$ has an intersection pairing $(\cdot,\cdot) = \int_{X^{[n]}} \cdot \cup \cdot$

Multiply it by $(-1)^{\dim X^{[n]}/2} = (-1)^n$

$$P_{-m}(\alpha) = P_m(\alpha)^* \qquad \text{adjoint}$$

Th (N, Grojnowski 1994)

1) $[P_m(\alpha), P_n(\beta)] = m\delta_{m+n,0}\,(\alpha,\beta)\,\mathrm{id}$ (Fock space of Heisenberg alg.)

(super comm. for odd α, β)

pairing on X (mult. by (-1))

2) (earlier by Göttsche)

$$\sum_n \dim H^*(X^{[n]})\, \mathfrak{z}^n = \text{character of Fock space}$$

i.e. $\bigoplus_n H^*(X^{[n]})$ is irreducible

- Proof is not so difficult, once we found the statement
  Except one nontrivial computation (determining the constant m),
  intersections are transversal on an open subset,
  and the complement does not contribute to the formula
  by dimension reason.

- discovery of the statement
  - my earlier result on affine Lie alg. $\rightsquigarrow$ quiver varieties
    $\leftarrow$ Lusztig's work
  - Vafa-Witten S-duality conjecture

§1-b, Virasoro algebra

$\{b_n\}$: Heisenberg alg. $\left(b_0 = a\,\mathrm{id} \quad (a \in \mathbb{C}) \quad (\text{center}) \right)$

$$b(z) := \sum_{n \in \mathbb{Z}} b_n z^{-n-1} \qquad (\text{field})$$

Let us set $T(z) := \frac{1}{2} :b(z)^2:$ $\left(:b_n b_m: = \begin{cases} b_n b_m & n \leq m \\ b_m b_n & n \geq m \end{cases} \right.$

$$T(z) := \frac{1}{2} :b(z)^2: = \sum L_n z^{-n-2}$$

$$\Longrightarrow \quad [L_m, L_n] = (m-n) L_{m+n} + \frac{m^3 - m}{12} \delta_{n+m,0}\, c \qquad \text{with} \quad c = 1$$

◎ Lehn ('98) gave a geometric realization of $T(z)$.
Let us explain this result.

<u>Fact</u> $X^{[n]}$ has a natural rank n vector b'dle $\mathcal{V}$, whose fiber at Z is

$$\mathcal{V}_Z = H^0(\mathcal{O}_Z)$$

<u>NB.</u> $c_1(\mathcal{V}) = -\frac{1}{2}[\partial X^{[n]}]$ $\qquad \partial X^{[n]} = \overline{\pi^{-1}(S_{(1^{n-1}2)})}$: divisor

coproduct $\Delta: H^*(X) \longrightarrow H^*(X) \otimes H^*(X) = -$(pushforward w.r.t. $\Delta: X \to X \times X$)

$$\Delta\alpha = \sum \alpha_{(1)} \otimes \alpha_{(2)} \quad \Rightarrow \quad :b_n b_m:(\alpha) \overset{\text{def.}}{=} \sum :b_n(\alpha_{(1)})\, b_m(\alpha_{(2)}):$$

Th [Lehn]

$$[G(v), P_n(\alpha)] = -n L_n(\alpha) + \frac{n(|n|-1)}{2} P_n(K\alpha)$$

where $K=$ canonical b'dle of the surface equivariant cohomology

$X = \mathbb{C}^2 \curvearrowleft T^2 = \mathbb{C}^* \times \mathbb{C}^*$ $\quad BT^2 = \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xleftarrow[T^2]{} ET^2 = (\mathbb{C}^\infty \setminus 0) \times (\mathbb{C}^\infty \setminus 0)$

$H^*_{T^2}(X^{[n]}) := H^*(X \times_{T^2} ET^2)$ (Borel construction)

$\curvearrowleft$ module over $H^*_{T^2}(pt) = \mathbb{C}[\varepsilon_1, \varepsilon_2]$

All the construction can be done in an equivariant way.
(Need to use equivariant homology)

— localized version : over $\mathrm{Frac}(H^*_{T^2}(pt)) = \mathbb{C}(\varepsilon_1, \varepsilon_2)$

$\Rightarrow \int$ is well-defined always $\leftarrow$ fixed pts sets are cpt

e.g. $\displaystyle \int_{\mathbb{C}^2} 1 = \frac{1}{\varepsilon_1 \varepsilon_2}$ (formal application of the fixed pt formula)

$K = -(\varepsilon_1 + \varepsilon_2)$

— nonlocalized version : over $\mathbb{C}[\varepsilon_1, \varepsilon_2]$

need to distinguish $H^*_{T^2,c}(X^{[n]})$ compact support

vs $H^*_{T^2}(X^{[n]})$ arbitrary support

e.g. $\bigoplus H^*_{T^2,c}(X^{[n]}) \curvearrowleft^{m>0}$ $P_m(\alpha)$ $\quad \alpha \in H^*_{T^2}(X)$

$P_{-m}(\alpha)$ $\quad \alpha \in H^*_{T^2,c}(X)$